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EQUILIBRIUM CONFIGURATION OF A PLASMA
IN THE GUIDING CENTER LIMIT

Alan Oppenheim

September 15, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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Institute of Mathematical Sciences
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Physics and
Mathematics

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Abstract

We compute the equilibrium configuration of a collision-free plasma contained in an axially symmetric magnetic field. The plasma is characterized by a non-scalar pressure tensor which is obtained from a microscopic distribution function in a form suggested by the guiding center approximation. The solution is calculated in the limit where the ratio of the width to the length of the plasma region and the ratio of the gas to the magnetic pressure are both small. Boundary values at the midplane as well as the shape of the plasma appear as arbitrary parameters in the solution. We give the solution to a corresponding scalar pressure problem for comparison.

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EQUILIBRIUM CONFIGURATION OF A PLASMA IN THE GUIDING CENTER LIMIT

1. Introduction

We present a calculation of the equilibrium configuration of a plasma in a magnetic field for a plasma with a non-scalar pressure tensor. Certain other studies of equilibrium consider scalar pressure plasmas (for example reference [1]) as is appropriate for a collision dominated fluid. Yet whenever it is appropriate to characterize the plasma as a collision-free gas, we expect that the presence of a magnetic field can be related to an anisotropy in the pressure tensor. For such a plasma the anisotropy will persist since there are no collisions to disturb it.

In Section 2 we assume axial symmetry and write the equilibrium equations in a form which takes advantage of this symmetry. We have an additional symmetry which follows from the guiding center approximation: the pressure tensor is axially symmetric about the magnetic field lines. That is, in a coordinate system with the z-axis tangent to the magnetic line, the pressure tensor P is diagonal and has the form (see reference [2])

$$(1) \quad P = \begin{pmatrix} P_{\perp} & 0 & 0 \\ 0 & P_{\perp} & 0 \\ 0 & 0 & P_{\parallel} \end{pmatrix},$$

where $P_{\perp}$ and $P_{\parallel}$ are called the components of P perpendicular and parallel to the direction of the magnetic line. In general $P_{\perp}$ is not equal to $P_{\parallel}$.

The equilibrium equations are:

$$(2) \quad \left\{ \begin{array}{l} \nabla \cdot \mathbf{P} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} \\ \nabla \cdot \vec{B} = 0 \end{array} \right.$$

where P is given by equation (1), $\vec{B}$ is the magnetic field, and μ_0 is the vacuum permeability. We need an additional relation to make this system of equations mathematically determinate since there are four equations (2) and five unknowns: $P_{\perp}$, $P_{\parallel}$ and the three components of $\vec{B}$. In Section 3 the additional relation, similar to an equation of state, is gotten from a microscopic distribution function. The general form of this distribution function is derived from the guiding center approximation. We ensure the validity of the approximation by requiring that the magnetic field should vary slowly with position. In addition we choose a specific form of distribution function which is useful for mirror geometries.*

* R. F. Post (reference [3]) has reported a calculation which is somewhat similar to ours. He does not use a distribution function but instead chooses a form for the magnetic field which approximates some experimental cases. He then solves for $P_{\perp}$.

Section 4 contains our solution to the equilibrium equations (2). We make two simplifying assumptions which enable us to exhibit the solution. The first assumption is that the ratio of the width to the length of the plasma region (the aspect ratio) is much smaller than unity. This assumption serves to remove the magnetic field line curvature from consideration and to reduce a partial differential equation to an ordinary differential equation. The second assumption is that the ratio of the plasma pressure to the magnetic pressure is much less than unity. This means that the presence of the plasma is a small perturbation on the vacuum solution for the magnetic field. We may then use an iterative technique to solve the equations. We find that if we prescribe a linear relation between P_1 and the magnetic flux at the midplane, as well as several other standard boundary conditions, then we get an explicit solution to the equilibrium equations. The other boundary conditions (or prescribed values) are: the prescription of the plasma boundary itself, the value of P_1 at the origin and of $\vec{B}$ at the boundary of the midplane. These three conditions (or values), subject to the two assumptions above, are otherwise arbitrary parameters in the solution. The final solution is then compared with the solution to the corresponding scalar pressure problem.

2. Equilibrium Equations for Axial Symmetry.

Since we deal only with axially symmetric configurations, it is useful to introduce the cylindrical coordinates (r, θ, z) . (See Figure 1 for this discussion.)

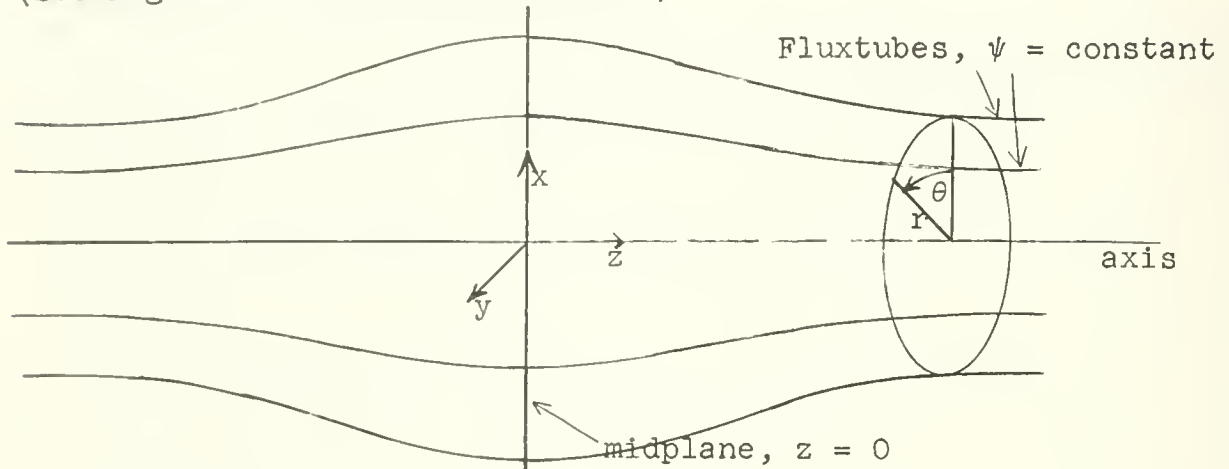


Figure 1.

We assume that there are no currents in the z -direction in the equilibrium state. This implies that B_θ vanishes throughout so that it is possible to introduce a stream function from which, by differentiation, we obtain B_r and B_z . Such a stream function is the magnetic flux ψ defined by

$$(3) \quad \psi(r, z) = 2\pi \int_0^r \rho d\rho B_z(\rho, z).$$

In cylindrical coordinates, the second of equations (2) is given by

$$(4) \quad \nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial}{\partial z} B_z = 0.$$

From (3) and (4) it follows that

$$(5) \quad B_r(r, z) = - \frac{1}{2\pi r} \frac{\partial \psi}{\partial z} ,$$

and

$$(6) \quad B_z(r, z) = \frac{1}{2\pi r} \frac{\partial \psi}{\partial r} .$$

From (5) and (6) we obtain

$$(7) \quad (\nabla \times \vec{B}) \times \vec{B} = - \left(\frac{1}{2\pi r} \right)^2 (L\psi) (\nabla \psi)$$

where

$$(8) \quad L\psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} .$$

We note that $(\nabla \times \vec{B}) \times \vec{B}$ lies in the direction of $\nabla \psi$. It seems natural then to evaluate $\nabla \cdot P$ in terms of coordinates along $\nabla \psi$ and $\vec{B}$. The $\vec{B}$ -lines lie on surfaces of constant ψ (flux tubes) so that our coordinate system actually consists of a grid of the surfaces of constant ψ and the orthogonal trajectories to these surfaces. We have already assumed that P has the form given by equation (1). In any other rectangular coordinate system (x_1, x_2, x_3) , the components of P are given by

$$(9) \quad P_{ij} = (P_{||} - P_{\perp}) \lambda_i \lambda_j + P_{\perp} \delta_{ij} ,$$

where

$$\vec{\lambda} = \frac{\vec{B}}{|\vec{B}|} ,$$

and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

We shall use the coordinate s , where s is length along a $\vec{B}$ -line, and write

$$\lambda_j \frac{\partial}{\partial x_j} = \frac{\partial}{\partial s}$$

and

$$\frac{\partial \vec{\lambda}}{\partial s} = \kappa(s) \vec{n}$$

where $\kappa(s)$ is the curvature of the $\vec{B}$ -line, and $\vec{n} = \nabla \psi / |\nabla \psi|$ is the principal normal to the $\vec{B}$ -line.

We get

$$(10) \quad \nabla \cdot P = \left\{ \frac{\partial}{\partial s} (P_{||} - P_{\perp}) - (P_{||} - P_{\perp}) \frac{\partial \log B}{\partial s} \right\} \vec{\lambda} \\ + (P_{||} - P_{\perp}) \kappa(s) \vec{n} + \nabla P_{\perp} ,$$

where $B = |\vec{B}|$.

Inserting (7) and (10) into the first of equations (2) we obtain the two equations:

$$(11) \quad \frac{\partial P_{\parallel}}{\partial s} - (P_{\parallel} - P_{\perp}) \frac{\partial \log B}{\partial s} = 0$$

and

$$(12) \quad \frac{\chi(s)}{|\nabla\psi|} (P_{\parallel} - P_{\perp}) + \frac{\nabla\psi \cdot \nabla P_{\perp}}{|\nabla\psi|^2} = - \frac{L\psi}{(2\pi r)^2 \mu_0}.$$

We can write equation (11) in the form

$$(13) \quad \frac{dP_{\parallel}}{d \log B} = P_{\parallel} - P_{\perp},$$

where the indicated differentiation is along a field line.

We have to solve the two equations, (11) and (12), for the three unknowns: $P_{\parallel}$, $P_{\perp}$ and ψ . This system of equations is obviously not determinate so that we must find an additional relation connecting the three unknowns. We do this in the following section by considering the individual particle motion in the guiding center limit.

3. The Form of the Pressure Tensor in the Guiding Center Limit.

In this section we find the additional relation linking $P_{\perp}$, $P_{\parallel}$ and ψ by considering the individual particle motion of the ions and electrons in the guiding center limit. This limit is that approximation (see reference [4]) in which the magnetic moment and the kinetic energy of a particle are constants of the motion as we follow the guiding center of the particle along a magnetic line. This means that ψ is also a constant of the motion since the magnetic lines lie on flux tubes (surfaces of constant ψ). The guiding center approximation follows from the restriction that the Larmor radius be smaller than a characteristic length of the system. We assume that $\vec{B}$ varies slowly with position so that the characteristic length is the maximum of $(B/|\nabla B|)$.

The following arguments apply to ions and electrons alike so that subscripts distinguishing the two species of particles are unnecessary. The equilibrium equations (2) combine the contributions of both species.

We assume the existence of a distribution function F for the particle density. The distribution may be written as a function of the constants of the motion. In the guiding center limit this motion is along the magnetic lines so that it is convenient to erect a local rectangular coordinate system on the magnetic line with the z -axis tangent to $\vec{B}$. In this notation, the particle velocity has a component $v_{\parallel}$ along

the z-axis and two components, designated by $\vec{v}_1$, in the plane normal to B. The three constants of the motion for a guiding center particle of mass m are the energy E, the magnetic moment μ , and the flux ψ where

$$(14) \quad E = \frac{m}{2} (v_1^2 + v_{||}^2),$$

$$(15) \quad \mu = \frac{mv_1^2}{2B},$$

and ψ identifies the magnetic line and is a function of position. The distribution function, $F(E, \mu, \psi)$, vanishes when $E < \mu B$ since, from (14) and (15), $E - \mu B = \frac{m}{2} v_{||}^2 \geq 0$.

So far $F(E, \mu, \psi)$ applies to any field configuration. We are actually interested in the case in which the plasma is confined by magnetic mirrors. The distribution will vanish for those values of E and μ which permit the particles to escape through the mirrors. That is: let B_m be the maximum value of B on a magnetic line, then particles with energies $E > \mu B_m$ will escape from the plasma region. In summary $F(E, \mu, \psi) = 0$ unless

$$(E/B_m) < \mu < (E/B).$$

Since there is no mass motion in our equilibrium configuration we can write the components of the pressure tensor in terms of F as follows:

$$(16) \quad P_{\parallel}(\psi) = m \int v_{\parallel}^2 F dV,$$

$$(17) \quad P_{\perp}(\psi) = \frac{m}{2} \int v_{\perp}^2 F dV,$$

where dV indicates integration over the three dimensional velocity space. In terms of E and μ these integrals become:

$$(18) \quad P_{\parallel}(\psi) = 2cB \int_0^{\infty} dE \int_{E/B_m}^{E/B} d\mu F(E, \mu, \psi) (E - \mu B)^{1/2}$$

$$(19) \quad P_{\perp}(\psi) = cB^2 \int_0^{\infty} dE \int_{E/B_m}^{E/B} \mu d\mu F(E, \mu, \psi) (E - \mu B)^{-1/2}$$

where $c = 2\pi(2m^3)^{-1/2}$. Following a particle path, we compute from (18) and (19) that

$$(20) \quad \frac{dP_{\parallel}}{d \log B} = P_{\parallel} - P_{\perp}.$$

Equation (20) is the same as equation (13) although the latter was derived from a macroscopic momentum balance. We remark that equation (20) does not depend on our restriction to mirror geometries. It is a general consequence of the guiding center approximation. Had we chosen to derive equation (20) before requiring that F vanish for $\mu B_m < E$, the result would have been the

same since the lower limit in the μ -integration in (18) and (19) would have been zero instead of E/B_m .

In general, the specification of F may be quite arbitrary. In any particular instance it may depend in a complicated way upon design parameters such as the manner of injection. One could, in principle, inject an arbitrary number of particles with given μ and E on each magnetic line. Each arbitrary specification of F determines $P_{||}$ and $P_{\perp}$ as functions of ψ . Among possible representations of F we might naturally consider terms of the form $f(E, \psi)(\mu B_m - E)^n$, which fall to zero properly as E approaches μB_m , or even a sum of such terms:

$$F(E, \mu, \psi) = \sum_{n=0}^{\infty} f_n(E, \psi)(\mu B_m - E)^n.$$

In order to obtain a fully worked out example we shall specify

$$(21) \quad F(E, \mu, \psi) = \begin{cases} f(E, \psi)(\mu B_m - E)^n, & \text{for } (E/B_m) < \mu < (E/B), \\ 0, & \text{otherwise,} \end{cases}$$

where $n \geq 0$ is not necessarily an integer.

With this choice, equations (18) and (19) become:

$$(22) \quad P_{||}(\psi) = A(\psi) \left(\frac{B_m}{B}\right)^n \left(1 - \frac{B}{B_m}\right)^{(n+3/2)},$$

$$(23) \quad P_{\perp}(\psi) = A(\psi) \left(\frac{B_m}{B}\right)^n \left(1 - \frac{B}{B_m}\right)^{(n+1/2)} \left((n+1) + \frac{B}{2B_m}\right),$$

where

$$A(\psi) = \frac{2c \Gamma(3/2) \Gamma(n+1)}{\Gamma(n+5/2)} \int_0^{\infty} f(E, \psi) E^{(n+3/2)} dE$$

and $\Gamma(x)$ denotes the gamma function.*

In (22) and (23), only $A(\psi)$ depends on whether we have electrons or ions. We add the contributions of the two sets of particles to form a new $A(\psi)$ and assume henceforth that equations (22) and (23) refer to the total plasma.

Let

$$(24) \quad M(\psi, s) = B_m(\psi) / B(\psi, s),$$

then

$$(25) \quad \frac{P_{\perp}(\psi, s)}{P_{\parallel}(\psi, s)} = \frac{(n+1)M(\psi, s) + 1/2}{M(\psi, s) - 1}.$$

(The coordinate s will be omitted henceforth.) Equation (25) is the relation which was needed to make the system of equations (2) determinate. From (25) we see that $(P_{\perp}/P_{\parallel}) > (n+1)$.

* See for example formulas 481 and 482 on Page 62 in B. O. Peirce, A Short Table of Integrals, Third Revised Edition, Ginn and Company.

In solving the equilibrium equations we will prescribe $P_{\perp}$ as a function of ψ at the midplane $s = 0$. It is convenient to give $A(\psi)$ in terms of $P_{\perp 0}$ (the subscript 0 indicates values at the midplane) and obtain equations (22) and (23) in a more useful form. From (23) we have

$$(26) \quad A(\psi) = P_{\perp 0}(\psi) M_0^{-n} (1 - 1/M_0)^{-(n+1/2)} [(n+1) + 1/(2M_0)]^{-1}.$$

Substituting for $A(\psi)$ in (22) and (23) we get

$$(27) \quad P_{\perp}(\psi) = P_{\perp 0}(\psi) \left(\frac{M}{M_0} \right)^n \left(\frac{1 - 1/M}{1 - 1/M_0} \right)^{(n+1/2)} \left[\frac{(n+1) + 1/(2M)}{(n+1) + 1/(2M_0)} \right]$$

and

$$(28) \quad P_{\parallel}(\psi) = P_{\perp 0}(\psi) \left(\frac{M}{M_0} \right)^n (1 - 1/M)^{(n+3/2)} (1 - 1/M_0)^{-(n+1/2)} [(n+1) + 1/(2M_0)]^{-1}.$$

From these equations we may draw a few qualitative results. We see that $P_{\parallel}$ and $P_{\perp}$ decrease monotonically as B increases and finally vanish at the mirrors where B is a maximum. $P_{\parallel}$ vanishes faster than $P_{\perp}$ at the mirrors. If $P_{\perp 0}(\psi)$ should vanish for some $\psi = \bar{\psi}$, then $P_{\perp} = P_{\parallel} = 0$ on the entire flux tube $\bar{\psi}$. This is used as the natural way of defining the boundary of the plasma.

4. Equilibrium Configuration for Low Aspect Ratio and Low β

The problem given by the equilibrium equations, (12) and (13), and the relation (25) is mathematically determinate once suitable boundary conditions are given. The discussion in the last section showed that $P_{||}$ and $P_{\perp}$ already satisfy equation (13). This was a consequence of the guiding center approximation. We will solve the remaining equation (12) after making three simplifying assumptions. The first, which is not very restrictive, is that the system is symmetric about the midplane. The second is that the aspect ratio (the ratio of the width to the length of the plasma region) is much smaller than unity. This considerably simplifies equation (12). The third assumption is that β (a representative ratio of gas to magnetic pressure) is much smaller than unity. This means that we can treat the plasma as a small perturbation imposed on a vacuum magnetic field. Our solution which will be explicit will be given to first order in β .

We need suitable boundary conditions on ψ before we can solve the equilibrium equation. First we introduce the notation to be used in the rest of this section. Quantities at the midplane (see Figure 2) are given the subscript o ,

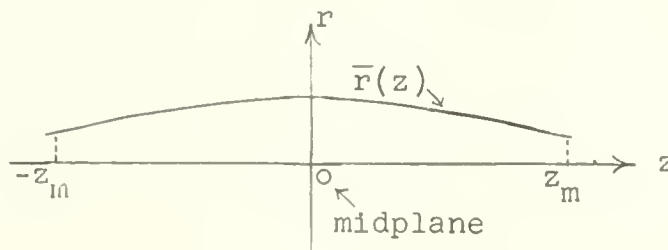


Figure 2.

those at the mirrors the subscript m . Quantities at the plasma boundary will have a bar over them, such as $\bar{\psi}$ or $\bar{B}$. In order to develop the boundary conditions we first prescribe the plasma boundary $\bar{r}(z)$. The only requirements on $\bar{r}(z)$ are that it vary slowly with z , that it have its maximum at the midplane ($z=0$), and that it decrease to a minimum at the mirrors ($z = \pm z_m$). The plasma boundary can be related to the coil system which is at our disposal. We next prescribe a linear relation between $P_{\perp}$ and ψ at the midplane:

$$(29) \quad P_{\perp 0}(\psi_0) = a \left\{ \psi_0(\bar{r}(0)) - \psi_0(r) \right\},$$

where a and $\psi_0(\bar{r}(0))$ are constants at our disposal. In addition we give $\bar{B}_0$ (B_z at the boundary on the midplane) and $P_{\perp 0}(0)$ ($P_{\perp 0}$ at the origin).

The choice (29) leads to an explicit solution at the midplane. Then we determine $\bar{\psi} = \psi_0(\bar{r}(0))$. This is one of the boundary conditions on $\psi(r,z)$. The other comes from equation (3) which implies that ψ vanishes on the axis. In summary, the boundary conditions on $\psi(r,z)$ are:

$$(30) \quad \left\{ \begin{array}{l} \psi(0,z) = 0 \\ \psi(\bar{r}(z),z) = \bar{\psi} \end{array} \right.$$

From (29) and (30) we see that $P_{10}(\psi)$ vanishes on $\bar{r}(z)$. This in turn implies that P_1 and P_{11} , given by (27) and (28), also vanish on the plasma boundary. Instead of prescribing $\bar{r}(z)$ we could just as well have prescribed $\vec{B}$ on the axis as an increasing function of z with maxima at the mirrors. We would then give the quantities $\bar{r}(0)$ and $P_{10}(0)$ and employ the new boundary conditions:

$$(30') \quad \left\{ \begin{array}{l} \psi(0, z) = 0 \\ \frac{1}{2\pi r} \frac{\partial \psi}{\partial r} \Big|_{r=0} = B_z(0, z). \end{array} \right.$$

The second equation in (30') follows from (6) and the fact that B_r vanishes on the axis. The two different sets of boundary conditions yield substantially the same solutions.

From (24) and (27) it is evident that we must first find $B_m(\psi)$, the maximum value of B attained at the mirrors, before we can solve in the interior. It turns out that B_m is a constant for all of the magnetic lines since the plasma vanishes at the mirrors.

We solve the equilibrium equation at arbitrary z to first order in β by a simple iteration process to be described at the appropriate place. This solution is then compared with the solution to the corresponding scalar pressure problem.

The detailed calculations now follow.

Low aspect ratio

We define the aspect ratio ϵ by

$$(31) \quad \epsilon = \bar{r}(0)/z_m$$

and assume that $\epsilon \ll 1$. This implies that all functions of interest may be written in the form

$$f = f(r, \epsilon z).$$

We neglect all terms of order ϵ^2 such as the field line curvature, second derivatives with respect to z , and products of first derivatives with respect to z . To this approximation equation (12) becomes

$$(32) \quad \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -4\pi^2 \mu_0 r^2 \frac{\partial P_{\perp}}{\partial r} / \frac{\partial \psi}{\partial r}.$$

We note that z enters only as a parameter in the differential equation. The formal solution to an equation of the form

$$(33) \quad \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = 4\pi^2 r^2 g(r)$$

is

$$(34) \quad \psi = 4\pi^2 \int_0^r t dt \int_0^t s g(s) ds + Cr^2,$$

where an additive constant has been set equal to zero since ψ vanishes on the axis. From our assumption that β is also much smaller than unity we shall show that $g(r)$ will be independent of r . For this case (34) reduces to

$$(24') \quad \psi = \pi^2 g \frac{r^4}{2} + Cr^2.$$

The constants will be determined from the boundary conditions as functions of z .

A further consequence of the assumption that ϵ be small is that B is equal to B_z to order ϵ . This follows from (5) since B_r is of order ϵ but appears only to order ϵ^2 in the magnitude of $\vec{B}$. We evaluate B by equation (35), henceforth:

$$(35) \quad B(r, z) = \frac{1}{2\pi r} \frac{\partial \psi}{\partial r}.$$

Solution at the midplane

To find the solution at the midplane we substitute (29) into the right hand side of (32) and obtain

$$(36) \quad \frac{d^2 \psi_0}{dr^2} - \frac{1}{r} \frac{d\psi_0}{dr} = 4\pi^2 r^2 \mu_0 a.$$

From equation (34') the solution is

$$(37) \quad \psi_0(r) = \pi^2 \mu_0 a \frac{r^4}{2} + Cr^2.$$

We prescribe $P_{\perp 0}(0)$ ($P_{\perp 0}$ at the origin) and $\bar{B}_0$ (B on the boundary of the midplane). The constants C and a are determined from (29) and (35).

$$(38) \quad C = \pi B_0(0),$$

and

$$(39) \quad a = \frac{B_0(0)}{\pi \mu_0 \bar{r}_0^2} [(1+\beta)^{1/2} - 1],$$

where

$$(40) \quad \beta = 2\mu_0 P_{\perp 0}(0) / B_0^2(0),$$

$$(41) \quad B_0(0) = (\bar{B}_0^2 - 2\mu_0 P_{\perp 0}(0))^{1/2},$$

and

$$\bar{r}_0 = \bar{r}(0)$$

Low β limit

The quantity β , defined by (40), is the maximum ratio of gas to magnetic pressure for the configuration we have chosen. We assume that $\beta \ll 1$, ignore terms of order β^2 , and obtain:

$$(39') \quad a = \left(\frac{B_o(0)}{2\pi\mu_o \bar{r}_o^2} \right) \beta,$$

$$(42) \quad \psi_o(r) = \pi B_o(0) r^2 \left\{ 1 + \left(\frac{r}{\bar{r}_o} \right)^2 \frac{\beta}{4} \right\}$$

and

$$(43) \quad \bar{\psi} = \psi_o(\bar{r}_o) = \pi B_o(0) \bar{r}_o^2 \left\{ 1 + \frac{\beta}{4} \right\}.$$

The boundary value $\bar{\psi}$ given by (43) is used in (30) to determine the solutions off the midplane.

For future use, we obtain from (35)

$$(44) \quad B_o(r) = B_o(0) \left\{ 1 + \left(\frac{r}{\bar{r}_o} \right)^2 \frac{\beta}{2} \right\}.$$

Solution at the mirrors.

We need $B_m (= B(r, z_m))$ in order to evaluate P_1 for $z \neq 0$. At the mirrors P vanishes so that we must solve (32) with the right hand side set equal to zero and impose the boundary conditions (30). We then determine B_m from (35) and obtain

$$(45) \quad B_m = \bar{\psi} / (\pi \bar{r}_m^2).$$

where

$$\bar{r}_m = \bar{r}(z_m).$$

The quantity B_m is a constant for all flux tubes to this approximation.

Solution off the midplane.

The right hand side of equation (32) is only of order β . Therefore it is reasonable to use an iteration scheme to solve this equation. The first step is to obtain the solution with the right hand side of (32) equal to zero and with the boundary conditions (30). From this solution, denoted by $\psi^{(0)}(r,z)$, we find $B^{(0)}(r,z)$ by means of (35). The next step is to insert $\psi^{(0)}$ and $B^{(0)}$ into the right hand side of (32) and solve for the next approximation to ψ which is denoted by $\psi^{(1)}(r,z)$. Further corrections will be of higher order in β so that we stop at $\psi^{(1)}$.

We need equation (27) in the more convenient form:

$$(46) \quad P_{\perp}(r,z) = P_{\perp 0}(\psi) \cdot h(r,z),$$

where

$$(47) \quad h(r,z) = \left(\frac{B_0(\psi)}{B(r,z)} \right)^n \cdot \left\{ \frac{1 - [B(r,z)/B_m]}{1 - [B_0(\psi)/B_m]} \right\}^{(n+1/2)} \cdot \left\{ \frac{(n+1) + [B(r,z)/2B_m]}{(n+1) + [B_0(\psi)/2B_m]} \right\}.$$

The notation $B_0(\psi)$ means that B_0 is evaluated at the mid-plane on the same flux tube ψ as the point (r,z) . The function $h(r,z)$ is unity at the midplane and decreases with increasing B to zero at the mirrors. We will find that the terms in r in $B(r,z)$ (just as for $B_0(r)$) are only of order β . The term on the right hand side of (32) is already of order β so that we need only evaluate h on the axis. This formal mathematical procedure is reasonable whenever the mirror ratio B_m/B_0 is not very close to unity. We insert $B^{(0)}$ and $\psi^{(0)}$ into (47) and get $h^{(0)}(0,z)$. The equation for $\psi^{(1)}(r,z)$ is

$$(48) \quad \frac{\partial^2 \psi^{(1)}}{\partial r^2} - \frac{1}{r} \frac{\partial \psi^{(1)}}{\partial r} = 4\pi^2 r^2 \mu_0 a h^{(0)}(0,z).$$

In order to find $\psi^{(0)}$ and $\psi^{(1)}$ we need only the formal solution (34') and the boundary conditions (30). Equation (35) then gives $B^{(0)}$ and $B^{(1)}$. The results are:

$$(49) \quad \psi^{(0)}(r,z) = \bar{\psi} \left(\frac{r}{\bar{r}(z)} \right)^2,$$

$$(50) \quad B^{(0)}(r,z) = B^{(0)}(z) = \bar{\psi} / (\pi \bar{r}^2(z))$$

$$(5) \quad h^{(0)}(0,z) = \left(\frac{\bar{r}(z)}{\bar{r}_0} \right)^{2n} \left\{ \frac{1 - [\bar{r}_m / \bar{r}(z)]^2}{1 - [\bar{r}_m / \bar{r}_0]^2} \right\}^{(n + \frac{1}{2})} \left\{ \frac{2(n+1) + [\bar{r}_m / \bar{r}(z)]^2}{2(n+1) + [\bar{r}_m / \bar{r}_0]^2} \right\},$$

$$(52) \quad \psi^{(1)}(r, z) = \psi^{(0)}(r, z) \left[1 + \frac{\beta}{4} h^{(0)}(0, z) \left(\frac{\bar{r}(z)}{\bar{r}_0} \right)^4 \left\{ \left(\frac{r}{\bar{r}(z)} \right)^2 - 1 \right\} \right]$$

$$(53) \quad B^{(1)}(r, z) = B^{(0)}(z) \left[1 + \frac{\beta}{4} h^{(0)}(0, z) \left(\frac{\bar{r}(z)}{\bar{r}_0} \right)^4 \left\{ 2 \left(\frac{r}{\bar{r}(z)} \right)^2 - 1 \right\} \right].$$

We could use (42), (44), (52) and (53) to evaluate $P_{\perp}$ numerically as a function of r and z . To lowest order in β , however, it is sufficient to use $\psi^{(0)}$ and $h^{(0)}(0, z)$ in (46). Using (29), (39'), (43), (46), (49) and (51) we find that

$$(54) \quad \frac{P_{\perp}(r, z)}{(B_0^2(0)/2\mu_0)} = \beta \left\{ 1 - \left(\frac{r}{\bar{r}(z)} \right)^2 \right\} h^{(0)}(0, z),$$

and from (25) that

$$(55) \quad \frac{P_{\perp}(r, z)}{(B_0^2(0)/2\mu_0)} = 2\beta \left\{ \frac{1 - (\bar{r}_m/\bar{r}(z))^2}{2(n+1) + (\bar{r}_m/\bar{r}(z))^2} \right\} \left\{ 1 - \left(\frac{r}{\bar{r}(z)} \right)^2 \right\} h^{(0)}(0, z).$$

Equations (52), (53), (54) and (55) (where $h^{(0)}$ is given in (51)) constitute our solution to the equilibrium equations to order β . The plasma boundary $\bar{r}(z)$ as well as $P_{\perp 0}(0)$ and $\bar{B}_0$ are arbitrary parameters in the solution, subject to the restrictions that $\epsilon \ll 1$ and that $\beta \ll 1$.

The corresponding scalar pressure problem.

In order that the scalar pressure problem correspond to the previous problem we preserve the same values at the midplane, the same plasma boundary $\bar{r}(z)$, and the same linear relation (29) between the scalar pressure P_s and ψ at the midplane. From equation (13) it follows that P_s is constant on a flux tube since $P_{||} = P_{\perp}$. We have therefore:

$$(56) \quad P_s = a(\bar{\psi} - \psi(r, z))$$

throughout the plasma region. Equation (32) becomes

$$(57) \quad \frac{\partial^2 \psi_s}{\partial r^2} - \frac{1}{r} \frac{\partial \psi_s}{\partial r} = 4\pi^2 r^2 \mu_0 a$$

where the subscript s denotes the scalar pressure quantities. We have the same boundary conditions (30) for this problem. The solution to (57) is given by (34'). We obtain

$$(58) \quad \psi_s = \bar{\psi} \left(\frac{r}{\bar{r}(z)} \right)^2 \left[1 + \frac{\beta}{4} \left(\frac{\bar{r}(z)}{\bar{r}_0} \right)^4 \left\{ \left(\frac{r}{\bar{r}(z)} \right)^2 - 1 \right\} \right],$$

$$(59) \quad B_s = \frac{\bar{\psi}}{\pi \bar{r}^2(z)} \left[1 + \frac{\beta}{4} \left(\frac{\bar{r}(z)}{\bar{r}_0} \right)^4 \left\{ 2 \left(\frac{r}{\bar{r}(z)} \right)^2 - 1 \right\} \right].$$

These solutions hold for arbitrary β although for the comparison we again take $\beta \ll 1$. To the same order of

approximation as before, the scalar pressure is given by

$$(60) \quad \frac{P_s}{(B_o^2(0)/2\mu_o)} = \beta \left\{ 1 - \left(\frac{r}{\bar{r}(z)} \right)^2 \right\} .$$

The comparison between $(\psi^{(1)}, B^{(1)})$ and (ψ_s, B_s) is not particularly revealing since they differ by a term of order β . In fact by setting $h^{(0)}(0,z)$ equal to unity in (52) and (53) we get the scalar pressure results (58) and (59). The function $h^{(0)}$ contains the essential difference between the scalar pressure and non-scalar pressure results.

Perhaps the more interesting comparison is the ratio $R = P_\perp/P_s$. From (54) and (60) we obtain

$$(61) \quad R = h^{(0)}(0,z).$$

This ratio starts out equal to unity at the midplane and decreases monotonically with increasing B until we reach the mirrors where R vanishes. This is to be expected since P_s remains constant on the flux tubes while $P_\parallel$ and $P_\perp$ both vanish at the mirrors.

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